

A Morita context and Galois extensions for Quasi-Hopf algebras

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Abstract

If H is a finite dimensional quasi-Hopf algebra and A is a left H -module algebra, we prove that there is a Morita context connecting the smash product $A\#H$ and the subalgebra of invariants A^H . We define also Galois extensions and prove the connection with this Morita context, as in the Hopf case.

1 Introduction

The Hopf Galois extensions appear for the first time in the papers of Chase, Harrison and Rosenberg ([10]) and of Chase and Sweedler ([11]). The actual definition is due to Kreimer and Takeuchi ([19]). A first Morita context has been constructed by Cohen, Fischman and Montgomery in [12]. They start from a finite dimensional Hopf algebra H over a field and an H -module algebra A , and construct a Morita context connecting the smash product $A\#H$ and the subalgebra of H -invariants A^H , showing also the connection with the Galois extensions. Another Morita context has been constructed by Doi, generalizing a construction of Chase and Sweedler ([11]). In this case, the author began with a Hopf algebra H not necessarily finite dimensional and an H -comodule algebra \mathcal{A} . The rings connected by the Morita context are this time \mathcal{B} , the subalgebra of coinvariants, and $\#(H, \mathcal{A})$.

In the case of a finite dimensional Hopf algebra H , a left H -module algebra is the same as a right H^* -comodule algebra, so it is natural to ask if both contexts coincide. The affirmative answer has been given by Beattie, Dăscălescu and Raianu in [2], for a co-Frobenius Hopf algebra.

The books of Montgomery [22] and of Dăscălescu, Năstăsescu and Raianu [13] are a good reference about the main results on this subject.

On the other hand, in the last fifteen years, several different generalizations of Hopf algebras have appeared: corings, weak Hopf algebras, quasi-Hopf algebras and Hopf algebroids. The Galois theory for corings was realized by Brzèzinski ([4]), while the Morita theory was developed for corings by Caenepeel, Vercruysse and Wang ([9]). The case of weak Hopf algebras and Hopf algebroids was considered by Böhm ([3]).

Quasi-Hopf algebras have been introduced by Drinfeld ([14]) and have lately attracted much attention in both mathematics and physics ([1], [20]). So it is desirable to see if it is possible to generalize the above results also to the case of quasi-Hopf algebras. This is the purpose of this paper.

If H is a Hopf algebra, to define a Galois extension, one usually starts with an H -comodule algebra \mathcal{A} and its subalgebra of coinvariants \mathcal{A}^{coH} . But the ordinary definition of coinvariants does not work any more in the quasi-Hopf setting. A possible approach was suggested in [23], but only in the case of a morphism $H \longrightarrow \mathcal{A}$ of right comodule algebras. If we turn to the finite dimensional case and work with module algebras, everything seems to be fitting. So we start with H a finite dimensional quasi-Hopf algebra, A a left H -module algebra and $B = A^H$ the subalgebra of invariants, which in this case is associative, while A is

not necessarily so. All the ingredients concerning quasi-Hopf actions were already defined in [7] and [8]. In order to get a Morita context between the smash product $A \# H$ and B , we need a right action of $A \# H$ on the linking bimodule A which is not obvious. The main ingredient in finding this action will be the **Remark 3.1.2**, which uses a formula for $S(t)$ (2.1.3), where $t \in H$ is a fixed nonzero left integral and S is the antipode of H .

Next, we define Galois extensions and, as in the Hopf case, we construct two canonical maps, showing that they are equivalent. A Frobenius-type isomorphism (**Proposition 2.1.5**) is used to show the equivalence between the canonical Galois map and one of the Morita maps.

In order to produce examples of Galois extensions, we remark first that this definition of Galois extensions is invariant to gauge transformations. Next, for \mathcal{A} a right H -comodule algebra (as it was defined in [15]), the quasi-smash product $\mathcal{A} \# H^*$ ([6]) is a Galois extension of \mathcal{A} , as in the Hopf case.

In the last part we study the surjectivity of the second Morita map, in connection with the notion of a total integral and the injectivity of relative modules.

The paper is ended by an analogue of Schneider's theorem in [24].

Our proofs will follow the original proofs of [12] and [2]. The main obstacle for the generalization is the comultiplication of H and the multiplication of A , which are no longer coassociative, respectively associative. These difficulties can be overcome by considering suitable elements that have been defined by Hausser and Nill ([15], [16], [17]) and their properties, which allow us to simplify the computations.

2 Preliminaries

In this section we recall some definitions and results and fix notations. Throughout the paper we work over some base field k . Tensor products, algebras, linear spaces, etc. will be over k . Unadorned \otimes means \otimes_k . An introduction to the study of quasi-bialgebras and quasi-Hopf algebras can be found in [14] or [18].

2.1 Quasi-Hopf Algebras and their integrals

Definition 2.1.1 A *quasi-bialgebra* means $(H, \Delta, \varepsilon, \phi)$ where H is an associative algebra with unit, ϕ is an invertible element in $H \otimes H \otimes H$ (the associator), $\Delta : H \longrightarrow H \otimes H$ (the coproduct) and $\varepsilon : H \longrightarrow k$ (the counit) are algebra homomorphisms, such that:

$$\phi(\Delta \otimes I)\Delta(h)\phi^{-1} = (I \otimes \Delta)\Delta(h) \quad (2.1a)$$

$$(I \otimes \varepsilon)\Delta(h) = (\varepsilon \otimes I)\Delta(h) = h \quad (2.1b)$$

$$(I \otimes I \otimes \Delta)(\phi)(\Delta \otimes I \otimes I)(\phi) = (1 \otimes \phi)(I \otimes \Delta \otimes I)(\phi)(\phi \otimes 1) \quad (2.1c)$$

$$(I \otimes \varepsilon \otimes I)(\phi) = 1 \otimes 1 \quad (2.1d)$$

hold for all $h \in H$.

The identities (2.1a)-(2.1d) also imply $(\varepsilon \otimes I \otimes I)(\phi) = (I \otimes I \otimes \varepsilon)(\phi) = 1 \otimes 1$.

As for Hopf algebras, we use the Sweedler's notation $\Delta(h) = h_1 \otimes h_2$, but since Δ is only quasi-coassociative we adopt the further convention:

$$(\Delta \otimes I)\Delta(h) = h_{1_1} \otimes h_{1_2} \otimes h_2 \text{ and } (I \otimes \Delta)\Delta(h) = h_1 \otimes h_{2_1} \otimes h_{2_2}$$

for all $h \in H$.

We shall denote the tensor components of ϕ by capital letters, and those of ϕ^{-1} by small letters, namely

$$\begin{aligned} \phi &= X^1 \otimes X^2 \otimes X^3 = Y^1 \otimes Y^2 \otimes Y^3 = \dots \\ \phi^{-1} &= x^1 \otimes x^2 \otimes x^3 = y^1 \otimes y^2 \otimes y^3 = \dots \end{aligned}$$

suppressing the summation symbol Σ .

Definition 2.1.2 A quasi-bialgebra is called a **quasi-Hopf algebra** if there is an anti-algebra homomorphism $S : H \longrightarrow H$ (the antipode) and elements $\alpha, \beta \in H$ such that

$$S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad (2.2a)$$

$$h_1\beta S(h_2) = \varepsilon(h)\beta \quad (2.2b)$$

$$X^1\beta S(X^2)\alpha X^3 = 1 \quad (2.2c)$$

$$S(x^1)\alpha x^2\beta S(x^3) = 1 \quad (2.2d)$$

hold for each $h \in H$.

The axioms for a quasi-Hopf algebra imply that $\varepsilon \circ S = \varepsilon$ and $\varepsilon(\alpha)\varepsilon(\beta) = 1$ so, by rescaling α and β , we may assume without loss of generality that $\varepsilon(\alpha) = \varepsilon(\beta) = 1$.

In this article we consider only finite dimensional quasi-Hopf algebra H . In this case the antipode of H is always bijective by [5].

Together with a quasi-Hopf algebra $H = (H, \Delta, \varepsilon, \phi, S, \alpha, \beta)$, we also have H^{op} , H^{cop} , and $H^{op,cop}$ as quasi-Hopf algebras, where "op" means opposite multiplication and "cop" means opposite comultiplication. The quasi-Hopf structures are obtained by putting $\phi_{op} = \phi^{-1}$, $\phi_{cop} = (\phi^{-1})^{321}$, $\phi_{op,cop} = \phi^{321}$, $S_{op} = S_{cop} = (S_{op,cop})^{-1} = S^{-1}$, $\alpha_{op} = S^{-1}(\beta)$, $\alpha_{cop} = S^{-1}(\alpha)$, $\alpha_{op,cop} = \beta$, $\beta_{op} = S^{-1}(\alpha)$, $\beta_{cop} = S^{-1}(\beta)$ and $\beta_{op,cop} = \alpha$.

Suppose that $(H, \Delta, \varepsilon, \phi)$ is a quasi-bialgebra. Then the category ${}_H\mathcal{M}$ of left H -modules is monoidal in the following way: for $U, V \in {}_H\mathcal{M}$ the tensor product $U \otimes V$ is an H -module by $h(u \otimes v) = h_1u \otimes h_2v$. The base field k is an H -module via ε . The canonical morphisms $U \simeq U \otimes k \simeq k \otimes U$ are H -linear for $U \in {}_H\mathcal{M}$. The map

$$\begin{aligned} \Phi_{U,V,W} & : (U \otimes V) \otimes W \longrightarrow U \otimes (V \otimes W) \\ u \otimes v \otimes w & \longrightarrow X^1u \otimes X^2v \otimes X^3w \end{aligned}$$

for $U, V, W \in {}_H\mathcal{M}$ is H -linear as a consequence of (2.1a), and satisfies Mac Lane's pentagon axiom for a monoidal category as a consequence of (2.1c).

In the Hopf algebra case, the antipode is an anti-coalgebra map. In order to have a similar property in the quasi-Hopf setting, Drinfeld ([14]) introduced a gauge element $f \in H \otimes H$, which obeys the following:

$$f\Delta(h)f^{(-1)} = (S \otimes S)\Delta^{cop}S^{-1}(h) \quad (2.3)$$

$$(S \otimes S \otimes S)(\phi^{321}) = (1 \otimes f)(I \otimes \Delta)(f)\phi(\Delta \otimes I)(f^{(-1)})(f^{(-1)} \otimes 1) \quad (2.4)$$

$$(I \otimes \varepsilon)(f) = (\varepsilon \otimes I)(f) = 1 \quad (2.5)$$

for all $h \in H$.

Following [15], [16], [17], we may define the elements

$$p_L = X^2S^{-1}(X^1\beta) \otimes X^3 \quad (2.6)$$

$$q_L = S(x^1)\alpha x^2 \otimes x^3 \quad (2.7)$$

$$p_R = x^1 \otimes x^2\beta S(x^3) \quad (2.8)$$

$$q_R = X^1 \otimes S^{-1}(\alpha X^3)X^2 \quad (2.9)$$

One may note that in H^{op} the roles of p_R and q_R (respectively p_L and q_L) interchange, and in H^{cop} one is passing from p_L to p_R (respectively from q_L to q_R), so whenever we have a relation concerning these

elements, it is enough to pass to H^{op} or to H^{cop} to get the other three relations. We shall state here only the relations used in this paper. Following [15], they obey the following

$$\Delta(h_2)p_L(S^{-1}(h_1) \otimes 1) = p_L(1 \otimes h) \quad (2.10)$$

$$(S(h_1) \otimes 1)q_L\Delta(h_2) = (1 \otimes h)q_L \quad (2.11)$$

$$\Delta(h_1)p_R(1 \otimes S(h_2)) = p_R(h \otimes 1) \quad (2.12)$$

$$(1 \otimes S^{-1}(h_2))q_R\Delta(h_1) = (h \otimes 1)q_R \quad (2.13)$$

for all $h \in H$ and

$$\Delta(q_L^2)p_L(S^{-1}(q_L^1) \otimes 1) = 1 \otimes 1 \quad (2.14)$$

$$(S(p_L^1) \otimes 1)q_L\Delta(p_L^2) = 1 \otimes 1 \quad (2.15)$$

$$\Delta(q_R^1)p_R(1 \otimes S(q_R^2)) = 1 \otimes 1 \quad (2.16)$$

$$(1 \otimes S^{-1}(p_R^2))q_R\Delta(p_R^1) = 1 \otimes 1 \quad (2.17)$$

$$\phi^{-1}(I \otimes \Delta)(p_L) = (\Delta(X^2)p_L \otimes X^3)(S^{-1}(X^1) \otimes 1 \otimes 1) \quad (2.18)$$

$$(I \otimes \Delta)(q_L)\phi = (S(x^1) \otimes 1 \otimes 1)(q_L\Delta(x^2) \otimes x^3) \quad (2.19)$$

$$\phi(\Delta \otimes I)(p_R) = (x^1 \otimes \Delta(x^2)p_R)(1 \otimes 1 \otimes S(x^3)) \quad (2.20)$$

$$(\Delta \otimes I)(q_R)\phi^{-1} = (1 \otimes 1 \otimes S^{-1}(X^3))(X^1 \otimes q_R\Delta(X^2)) \quad (2.21)$$

We shall need also the following elements:

$$V_L = (S \otimes S)(p_L^{21})f \quad (2.22)$$

$$U_L = (S^{-1} \otimes S^{-1})(q_L f^{(-1)})^{21} \quad (2.23)$$

$$V_R = (S^{-1} \otimes S^{-1})(f p_R)^{21} \quad (2.24)$$

$$U_R = f^{-1}(S \otimes S)(q_R^{21}) \quad (2.25)$$

U_R and V_R were introduced by Nill and Hausser in [17], and U_L and V_L are their analogues by passing from H to H^{cop} , as explained above. They obey the following relations:

$$(1 \otimes h_1)V_L\Delta S(h_2) = (S(h) \otimes 1)V_L \quad (2.26)$$

$$\Delta S^{-1}(h_2)U_L(1 \otimes h_1) = U_L(S^{-1}(h) \otimes 1) \quad (2.27)$$

$$(h_2 \otimes 1)V_R\Delta S^{-1}(h_1) = (1 \otimes S^{-1}(h))V_R \quad (2.28)$$

$$\Delta S(h_1)U_R(h_2 \otimes 1) = U_R(1 \otimes S(h)) \quad (2.29)$$

We need also to notice that H finite dimensional implies H^* is a coassociative coalgebra, with coproduct given by

$$\Delta^*(h^*) = h^* \circ \mu_H, \forall h^* \in H^*$$

The comultiplication Δ on H allows us to define a multiplication on H^* (the convolution product) by

$$(h^* g^*)(h) = h^*(h_1)g^*(h_2), \forall h^* \in H^*, g, h \in H$$

But this is no longer associative. In fact H^* , endowed with this multiplication, is an algebra in the monoidal category of H -bimodules and we have that

$$(h^* g^*)l^* = (X^1 \rightharpoonup h^* \leftharpoonup x^1)[(X^2 \leftharpoonup g^* \leftharpoonup x^2)(X^3 \rightharpoonup l^* \rightharpoonup x^3)]$$

where $h^*, g^*, l^* \in H^*$. By \leftarrow and \rightarrow we denote the usual right, respectively left H -action on H^* :

$$(h \rightarrow h^*)(g) = h^*(gh), (h^* \leftarrow h)(g) = h^*(hg)$$

for all $h^* \in H^*, g, h \in H$.

Although H^* is not an algebra, we still keep the notation of the Hopf case for the weak action of H^* on H :

$$h^* \rightarrow h = h^*(h_2)h_1, h \leftarrow h^* = h^*(h_1)h_2$$

for all $h^* \in H^*, h \in H$.

We denote by \int_H^l the space of left integrals in H and by $t \in H$ a nonzero left integral. As \int_H^l is an ideal of H (which is one dimensional by [17]), there is only one algebra morphism (the modular element) $\gamma \in H^*$ which satisfies

$$th = \gamma(h)t \tag{2.30}$$

for all $h \in H$. Denote by $\Lambda = \gamma(q_L^2)q_L^1$. The next result was proved by Bulacu and Caenepeel in [5], **Proposition 4.9** for a dual quasi-Hopf algebra. But if H is finite dimensional, H^* is a dual quasi-Hopf algebra, so we can restate their result in terms of quasi-Hopf algebras to get the following:

Proposition 2.1.3 *Let H be a finite dimensional quasi-Hopf algebra. With the above notation we have*

$$S(t) = \Lambda(\gamma \rightarrow t).$$

The following statement can be obtained from the same authors quoted above ([5], **Theorem 2.2**) by passing from H to H^{cop} :

Theorem 2.1.4 *Let H be a finite dimensional quasi-Hopf algebra, $(e_i)_{i=1,n}$ a basis of H and $(e^i)_{i=1,n}$ the dual basis of H^* . We define the map*

$$P : H \longrightarrow H, P(h) = \sum_{i=1}^n e^i(S^{-2}(q_L^1 e_{i1} S(\beta))h)q_L^2 e_{i2}$$

for all $h \in H$. Then:

1. $P(h) \in \int_H^l$, for all $h \in H$ and there is $h \in H$ such that $P(h) \neq 0$;
2. The map $\Theta : \int_H^l \otimes H^* \longrightarrow H$,

$$\Theta(t \otimes h^*) = h^*(q_L^1 t_1 p_L^1)q_L^2 t_2 p_L^2, \forall t \in \int_H^l, h^* \in H^*$$

is an isomorphism of left H -modules with inverse given by

$$\Theta^{-1}(h) = \sum_{i=1}^n P(e_i h) \otimes e^i S^{-1}, \forall h \in H,$$

where $\int_H^l \otimes H^*$ is a left H -module via $h(t \otimes h^*) = t \otimes h^* \leftarrow S(h), \forall h \in H, t \in \int_H^l, h^* \in H^*$.

This allows us to consider the following Frobenius-type isomorphism:

Corollary 2.1.5 *Let H be a finite dimensional quasi-Hopf algebra and $t \in \int_H^l$ a nonzero left integral. Then the map*

$$\theta_t : H^* \longrightarrow H, \theta_t(h^*) = h^*(q_L^1 t_1 p_L^1) q_L^2 t_2 p_L^2, \forall h^* \in H^*$$

is a left H -module isomorphism.

By [5] or [21], we have, for $t \in H$ a left integral and all $h \in H$:

$$(S(h) \otimes 1) q_L \Delta(t) = (1 \otimes h) q_L \Delta(t) \quad (2.31)$$

$$(S(h) \otimes 1) q_R \Delta(t) = (1 \otimes h) q_R \Delta(t) \quad (2.32)$$

and

$$\Delta(t) = (\beta \otimes 1) q_L \Delta(t) = (\beta \otimes 1) q_R \Delta(t) \quad (2.33)$$

$$= (1 \otimes S^{-1}(\beta)) q_L \Delta(t) = (1 \otimes S^{-1}(\beta)) q_R \Delta(t) \quad (2.34)$$

2.2 H -module algebras

Recall from [8] the notion of a module algebra over a quasi-bialgebra.

Definition 2.2.1 *Let H be a quasi-bialgebra and A a linear space. We say that A is a (left) H -**module algebra** if A is an algebra in the monoidal category ${}_H\mathcal{M}$, i.e. A is a left H -module which has a multiplication and a usual unit 1_A satisfying the following conditions:*

$$(ab)c = (X^1 \cdot a)[(X^2 \cdot b)(X^3 \cdot c)] \quad (2.35a)$$

$$h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b) \quad (2.35b)$$

$$h \cdot 1_A = \varepsilon(h) 1_A \quad (2.35c)$$

for all $a, b, c \in A$ and $h \in H$, where $h \otimes a \longrightarrow h \cdot a$ is the H -module structure of A .

For an H -module algebra A , we may define the **smash product** $A \# H$ as in [8]: as a vector space $A \# H$ is $A \otimes H$ with multiplication given by

$$(a \# h)(b \# g) = (x^1 \cdot a)(x^2 h_1 \cdot b) \# x^3 h_2 g \quad (2.36)$$

for all $a, b \in A$ and $g, h \in H$. Then $A \# H$ becomes an associative algebra with unit $1_A \# 1$. Also for the H -module algebra A we may define the subalgebra of invariants $B = A^H$, that is

$$B = \{a \in A \mid h \cdot a = \varepsilon(h)a, \forall h \in H\}$$

Remark that B is an associative algebra, and A is a left and right B -module in a natural way. Also, on A we have a left $A \# H$ -module structure given by

$$(a \# h)b = a(h \cdot b)$$

for all $a, b \in A$ and $h \in H$.

As A is an algebra in the monoidal category of left H -modules, it is natural to consider modules over A (left or right) in this category. These were called relative Hopf modules in [8]:

Definition 2.2.2 Let H be a quasi-bialgebra and A a left H -module algebra. A k -vector space M is called a left (H, A) -**Hopf module** if M is a left H -module and also a left A -module in the monoidal category ${}_H\mathcal{M}$, i.e. A acts on M to the left with action denoted $a \otimes m \longrightarrow am$ such that

$$(ab)m = (X^1 \cdot a)[(X^2 \cdot b)(X^3 m)] \quad (2.37a)$$

$$h(am) = (h_1 \cdot a)(h_2 m) \quad (2.37b)$$

$$1_A m = m \quad (2.37c)$$

for all $h \in H$, $a, b \in A$ and $m \in M$.

The category of left (H, A) -Hopf modules with morphisms that are left H -linear and preserve the weak A -action will be denoted by ${}_A(H\mathcal{M})$.

In [8] it is proved that the categories ${}_A(H\mathcal{M})$ and ${}_{A\#H}\mathcal{M}$ are isomorphic: if $M \in {}_A(H\mathcal{M})$, then $M \in {}_{A\#H}\mathcal{M}$ by $(a\#h)m = a(hm)$, and if $M \in {}_{A\#H}\mathcal{M}$ then $M \in {}_A(H\mathcal{M})$ by $am = (a\#1)m$ and $hm = (1_A\#h)m$, where $h \in H$, $a \in A$ and $m \in M$.

Now for each $M \in {}_A(H\mathcal{M})$, denote $M^H = \{m \in M | hm = \varepsilon(h)m, \forall h \in H\}$. Then M^H becomes naturally a left B -module, so we get a functor

$${}_A(H\mathcal{M}) \xrightarrow{(-)^H} {}_B\mathcal{M}$$

Conversely, for $M \in {}_B\mathcal{M}$, we have $A \otimes_B M \in {}_A(H\mathcal{M})$ by

$$h(a \otimes_B m) = h \cdot a \otimes_B m$$

$$b(a \otimes_B m) = ba \otimes_B m$$

As in the classical Hopf algebra case, we obtain the following:

Proposition 2.2.3 The induced functor $A \otimes_B (-)$ is a left adjoint for the functor of invariants $(-)^H$

$${}_B\mathcal{M} \begin{matrix} \xrightarrow{A \otimes_B (-)} \\ \xleftarrow{(-)^H} \end{matrix} {}_A(H\mathcal{M})$$

Proof. The adjunction morphisms are:

$$\begin{aligned} f \in \text{Hom}_{{}_A(H\mathcal{M})}(A \otimes_B M, N) & \xrightleftharpoons[\beta]{\alpha} \text{Hom}_B(M, N^H) \ni g \\ \alpha(f)(m) &= f(1_A \otimes_B m) \\ \beta(g)(a \otimes_B m) &= ag(m) \end{aligned}$$

for all $m \in M$, $a \in A$. ■

Remark 2.2.4 As ${}_A(H\mathcal{M})$ and ${}_{A\#H}\mathcal{M}$ are isomorphic, we get also the adjunction ${}_B\mathcal{M} \begin{matrix} \xrightarrow{A \otimes_B (-)} \\ \xleftarrow{(-)^H} \end{matrix} {}_{A\#H}\mathcal{M}$.

3 The Morita context

3.1 Construction

In this section we construct a Morita context connecting $A\#H$ and $B = A^H$, where A is our left H -module algebra. By [8], **Proposition 2.7**, the map $H \longrightarrow A\#H$, $h \longrightarrow 1_A\#h$ is an algebra morphism. It induces a structure of left H -module on $A\#H$ by

$$h \times (a\#g) = (1_A\#h)(a\#g) = h_1 \cdot a\#h_2 g \quad (3.1)$$

for all $a \in A$ and $g, h \in H$. Also, the tensor product $H \otimes A$ has the structure of a left H -module with action induced by the multiplication of H :

$$h * (g \otimes a) = hg \otimes a$$

for all $a \in A$ and $g, h \in H$.

Proposition 3.1.1 *Under the above assumptions, the map $\varphi : H \otimes A \longrightarrow A \# H$, $\varphi(h \otimes a) = h_1 p_L^1 \cdot a \# h_2 p_L^2$ is an isomorphism of left H -modules, with inverse $\varphi^{-1}(a \# h) = q_L^2 h_2 \otimes S^{-1}(q_L^1 h_1) \cdot a$.*

Proof. We have

$$\begin{aligned} \varphi(h * (g \otimes a)) &= \varphi(hg \otimes a) \\ &= h_1 g_1 p_L^1 \cdot a \# h_2 g_2 p_L^2 \\ (3.1) \quad &= h \times (g_1 p_L^1 \cdot a \# g_2 p_L^2) \\ &= h \times \varphi(g \otimes a) \end{aligned}$$

for all $a \in A$ and $g, h \in H$, so φ is H -linear. Let's check that φ and φ^{-1} are inverses to each other:

$$\begin{aligned} \varphi\varphi^{-1}(a \# h) &= \varphi(q_L^2 h_2 \otimes S^{-1}(q_L^1 h_1) \cdot a) \\ &= q_{L1}^2 h_{21} p_L^1 S^{-1}(q_L^1 h_1) \cdot a \# q_{L2}^2 h_{22} p_L^2 \\ (2.10) \quad &= q_{L1}^2 p_L^1 S^{-1}(q_L^1) \cdot a \# q_{L2}^2 p_L^2 h \\ (2.14) \quad &= a \# h \end{aligned}$$

and

$$\begin{aligned} \varphi^{-1}\varphi(h \otimes a) &= \varphi^{-1}(h_1 p_L^1 \cdot a \# h_2 p_L^2) \\ &= q_L^2 h_{22} p_{L2}^2 \otimes S^{-1}(q_L^1 h_{21} p_{L1}^2) h_1 p_L^1 \cdot a \\ (2.11) \quad &= h q_L^2 p_{L2}^2 \otimes S^{-1}(q_L^1 p_{L1}^2) p_L^1 \cdot a \\ (2.15) \quad &= h \otimes a \end{aligned}$$

for all $a \in A$ and $h \in H$. ■

Remark 3.1.2 *Restricting the above isomorphism, we get an isomorphism of H -invariants: $(A \# H)^H \simeq (H \otimes A)^H$. But H is acting on $H \otimes A$ by left multiplication on the first component, so $(H \otimes A)^H = H^H \otimes A = \int_H^l \otimes A \simeq A$. On the other side, $A \# H$ is a right $A \# H$ -module with action induced by multiplication, and the restriction $(A \# H)^H$ remains a right $A \# H$ -module, because for $a \# h \in (A \# H)^H$, $b \# g \in A \# H$ and $l \in H$ we have*

$$\begin{aligned} l \times ((a \# h)(b \# g)) &= (1 \# l)[(a \# h)(b \# g)] \\ &= [(1 \# l)(a \# h)](b \# g) \\ &= \varepsilon(l)(a \# h)(b \# g) \end{aligned}$$

Hence, the isomorphism of **Proposition 3.1.1** induces a structure of right $A \# H$ -module on $A \simeq \int_H^l \otimes A$. Explicitly, this means

$$\begin{aligned} a \otimes (b \# h) &\longrightarrow (t \otimes a) \otimes (b \# h) \xrightarrow{\varphi \otimes I} (t_1 p_L^1 \cdot a \# t_2 p_L^2) \otimes (b \# h) \\ &\longrightarrow (t_1 p_L^1 \cdot a \# t_2 p_L^2)(b \# h) \\ &= (x^1 t_1 p_L^1 \cdot a)(x^2 t_{21} p_{L1}^2 \cdot b) \# x^3 t_{22} p_{L2}^2 h \end{aligned}$$

$$\begin{aligned}
(2.1a) &= (t_1 x^1 p_L^1 \cdot a)(t_1 x^2 p_{L1}^2 \cdot b) \# t_2 x^3 p_{L2}^2 h \\
(2.18) &= t_1 X^2 \cdot [(p_L^1 S^{-1}(X^1) \cdot a)(p_L^2 \cdot b)] \# t_2 X^3 h \\
&\xrightarrow{\varphi^{-1}} q_L^2 t_{22} X_2^3 h_2 \otimes S^{-1}(q_L^1 t_{21} X_1^3 h_1) t_1 X^2 \\
&\quad \cdot [(p_L^1 S^{-1}(X^1) \cdot a)(p_L^2 \cdot b)] \\
(2.11) &= t q_L^2 X_2^3 h_2 \otimes S^{-1}(q_L^1 X_1^3 h_1) X^2 \cdot [(p_L^1 S^{-1}(X^1) \cdot a)(p_L^2 \cdot b)] \\
(2.30) &= t \otimes S^{-1}(q_L^1 X_1^3 h_1) X^2 \cdot [(p_L^1 S^{-1}(X^1) \cdot a)(p_L^2 \cdot b)] \gamma(q_L^2 X_2^3 h_2) \\
&= t \otimes S^{-1}(S(X^2) \Lambda(\gamma \rightharpoonup X^3 h)) \cdot [(p_L^1 S^{-1}(X^1) \cdot a)(p_L^2 \cdot b)]
\end{aligned}$$

where we denoted $\Lambda = q_L^1 \gamma(q_L^2)$. Therefore, the right $A \# H$ -module structure on A is given by

$$a \cdot_\gamma (b \# h) = S^{-1}(S(X^2) \Lambda(\gamma \rightharpoonup X^3 h)) \cdot [(p_L^1 S^{-1}(X^1) \cdot a)(p_L^2 \cdot b)] \quad (3.2)$$

This can be also checked directly, but very long calculations are involved.

By (3.2), A becomes a $(B, A \# H)$ -bimodule. In the previous section we endowed A with an $(A \# H, B)$ -bimodule structure, so now we may pass to the next step:

Theorem 3.1.3 Consider the maps:

$$(-, -) : A \otimes_{A \# H} A \longrightarrow B, (a, b) = t \cdot [(p_L^1 \cdot a)(p_L^2 \cdot b)] \quad (3.3)$$

$$[-, -] : A \otimes_B A \longrightarrow A \# H, [a, b] = (a \# t)(p_L^1 \cdot b \# p_L^2) \quad (3.4)$$

Then $(A \# H, B, {}_{A \# H} A_B, {}_B A_{A \# H}, (-, -), [-, -])$ is a Morita context.

Proof. We check only the conditions of a Morita context which are more difficult because of the $A \# H$ -module structures involved, the others are left to the reader:

- $(-, -)$ is well-defined ($A \# H$ -balanced):

$$\begin{aligned}
(a \cdot_\gamma (b \# h), c) &= (S^{-1}(S(X^2) \Lambda(\gamma \rightharpoonup X^3 h)) \cdot [(p_L^1 S^{-1}(X^1) \cdot a)(p_L^2 \cdot b)], c) \\
(3.3) &= t \cdot \{(P_L^1 S^{-1}(S(X^2) \Lambda(\gamma \rightharpoonup X^3 h)) \cdot [(p_L^1 S^{-1}(X^1) \cdot a)(p_L^2 \cdot b)])(P_L^2 \cdot c)\} \\
(2.30) &= t q_L^2 X_2^3 h_2 \cdot \{(P_L^1 S^{-1}(q_L^1 X_1^3 h_1) X^2 \cdot [(p_L^1 S^{-1}(X^1) \cdot a)(p_L^2 \cdot b)])(P_L^2 \cdot c)\} \\
(2.10) &= t q_L^2 \cdot \{(P_L^1 S^{-1}(q_L^1) X^2 \cdot [(p_L^1 S^{-1}(X^1) \cdot a)(p_L^2 \cdot b)])(P_L^2 X^3 h \cdot c)\} \\
(2.14) &= t \cdot \{X^2 \cdot [(p_L^1 S^{-1}(X^1) \cdot a)(p_L^2 \cdot b)](X^3 h \cdot c)\} \\
(2.18) &= t \cdot \{(x^1 p_L^1 \cdot a)(x^2 p_{L1}^2 \cdot b)(x^3 p_{L2}^2 h \cdot c)\} \\
(2.35a), (2.35b) &= t \cdot \{(p_L^1 \cdot a)[p_L^2 \cdot (b(h \cdot c))]\} \\
(3.3) &= (a, b(h \cdot c)) \\
&= (a, (b \# h)c)
\end{aligned}$$

for each $a, b, c \in A$ and $h \in H$, where $P_L^1 \otimes P_L^2$ is another copy of p_L .

- $[-, -]$ is $A \# H$ -bilinear: for the left linearity, we compute that

$$\begin{aligned}
[(a \# h)b, c] &= [a(hb), c] \\
(3.4) &= (a(hb) \# t)(p_L^1 c \# p_L^2) \\
&= [(x^1 a)(x^2 h_1 b) \# x^3 h_2 t](p_L^1 c \# p_L^2) \\
(2.36) &= (a \# h)(b \# t)(p_L^1 c \# p_L^2) \\
&= (a \# h)[b, c]
\end{aligned}$$

where in the third line we used the fact that t is a left integral. For the right $A\#H$ -linearity it's a little more complicated because of the right action of $A\#H$ on A :

$$\begin{aligned}
[a, b \cdot_\gamma (c\#h)] &= [a, S^{-1}(S(X^2)\Lambda(\gamma \rightharpoonup X^3h)) \cdot [(p_L^1 S^{-1}(X^1) \cdot b)(p_L^2 \cdot c)]] \\
(3.4) &= (a\#t)(P_L^1 S^{-1}(S(X^2)\Lambda(\gamma \rightharpoonup X^3h)) \cdot [(p_L^1 S^{-1}(X^1) \cdot b)(p_L^2 \cdot c)] \# P_L^2) \\
(2.30) &= (a\#t q_L^2 X_2^3 h_2)(P_L^1 S^{-1}(q_L^1 X_1^3 h_1) X^2 \cdot [(p_L^1 S^{-1}(X^1) \cdot b)(p_L^2 \cdot c)] \# P_L^2) \\
(2.36) &= (x^1 \cdot a)(x^2 t_1 q_{L1}^2 X_{21}^3 h_{21} P_L^1 S^{-1}(q_L^1 X_1^3 h_1) X^2 \cdot [(p_L^1 S^{-1}(X^1) \cdot b)(p_L^2 \cdot c)]) \\
&\quad \# x^3 t_2 q_{L2}^2 X_{22}^3 h_{21} P_L^2 \\
(2.10) &= (x^1 \cdot a)(x^2 t_1 q_{L1}^2 P_L^1 S^{-1}(q_L^1) X^2 \cdot [(p_L^1 S^{-1}(X^1) \cdot b)(p_L^2 \cdot c)]) \\
&\quad \# x^3 t_2 q_{L2}^2 P_L^2 X^3 h \\
(2.14) &= (x^1 \cdot a)(x^2 t_1 X^2 \cdot [(p_L^1 S^{-1}(X^1) \cdot b)(p_L^2 \cdot c)]) \# x^3 t_2 X^3 h \\
(2.18) &= (x^1 \cdot a)(x^2 t_1 \cdot [(y^1 p_L^1 \cdot b)(y^2 p_{L1}^2 \cdot c)] \# x^3 t_2 y^3 p_{L2}^2 h \\
(2.36) &= (a\#t)((y^1 p_L^1 \cdot b)(y^2 p_{L1}^2 \cdot c) \# y^3 p_{L2}^2 h) \\
(2.36) &= (a\#t)(p_L^1 \cdot b \# p_L^2)(c\#h) \\
(3.4) &= [a, b](c\#h)
\end{aligned}$$

for all $a, b, c \in A$ and $h \in H$, where $P_L^1 \otimes P_L^2$ is another copy of p_L .

- associativity of the Morita map:

$$\begin{aligned}
a \cdot_\gamma [b, c] &= a \cdot_\gamma [(b\#t)(p_L^1 \cdot c\#p_L^2)] \\
&= [a \cdot_\gamma (b\#t)] \cdot_\gamma (p_L^1 \cdot c\#p_L^2) \\
(3.2) &= \{S^{-1}(S(X^2)\Lambda(\gamma \rightharpoonup X^3t)) \cdot [(P_L^1 S^{-1}(X^1) \cdot a)(P_L^2 \cdot b)]\} \cdot_\gamma (p_L^1 c\#p_L^2) \\
&= \{S^{-1}(\Lambda(\gamma \rightharpoonup t)) \cdot [(P_L^1 \cdot a)(P_L^2 \cdot b)]\} \cdot_\gamma (p_L^1 c\#p_L^2) \\
(2.1.3) &= \{S^{-1}(S(t)) \cdot [(P_L^1 \cdot a)(P_L^2 \cdot b)]\} \cdot_\gamma (p_L^1 \cdot c\#p_L^2) \\
&= \{t \cdot [(P_L^1 \cdot a)(P_L^2 \cdot b)]\} \cdot_\gamma (p_L^1 \cdot c\#p_L^2) \\
&= (a, b) \cdot_\gamma (p_L^1 \cdot c\#p_L^2) \\
(3.2) &= S^{-1}(S(X^2)\Lambda(\gamma \rightharpoonup X^3p_L^2)) \cdot [(P_L^1 S^{-1}(X^1) \cdot (a, b))(P_L^2 p_L^1 \cdot c)] \\
&= S^{-1}(\Lambda(\gamma \rightharpoonup p_L^2)) \cdot [(a, b)(p_L^1 \cdot c)] \\
&= S^{-1}(\Lambda(\gamma \rightharpoonup p_L^2)) \cdot [\varepsilon(p_{L1}^1)(a, b)(p_{L2}^1 \cdot c)] \\
&= S^{-1}(\Lambda(\gamma \rightharpoonup p_L^2)) \cdot [(p_{L1}^1 \cdot (a, b))(p_{L2}^1 \cdot c)] \\
&= S^{-1}(\Lambda(\gamma \rightharpoonup p_L^2)) p_L^1 \cdot ((a, b)c) \\
&= S^{-1}(q_L^1 p_{L1}^2) p_L^1 \cdot ((a, b)c) \gamma(q_L^2 p_{L2}^2) \\
&= (a, b)c
\end{aligned}$$

for all $a, b, c \in A$, where in the fourth line we used the fact that t is a left integral and in the following lines that $(a, b) \in A^H$.

■

3.2 Surjectivity of the Morita maps

3.2.1 Galois Extensions

Definition 3.2.1 Let H be a finite dimensional quasi-Hopf algebra, $\dim_k H = n$, A a left H -module algebra and $B = A^H$ the subalgebra of H -invariants. We say that the extension $B \subseteq A$ is **Galois** if the linear map $\text{can} : A \otimes_B A \longrightarrow A \otimes H^*$, $\text{can}(a \otimes_B b) = \sum_{i=1}^n (p_R^1 \cdot a)(p_R^2 e_i \cdot b) \otimes e^i$ is bijective, where $(e_i)_{i=1,n}$ and $(e^i)_{i=1,n}$ are dual bases for H , respectively H^* .

Remark 3.2.2 As in the Hopf case, one may instead use the map $\text{can}' : A \otimes_B A \longrightarrow A \otimes H^*$, $\text{can}'(a \otimes_B b) = \sum_{i=1}^n (U_R^1 e_i \cdot a)(U_R^2 \cdot b) \otimes e^i$ where U_R is the element from (2.25). If we define

$$\Xi : A \otimes H^* \longrightarrow A \otimes H^*, \Xi(a \otimes h^*) = \sum_{i=1}^n e_i \cdot a \otimes (e^i \leftarrow q_L^1)(h^* S \leftarrow q_L^2)$$

then we have the connection between the two "can" maps: $\Xi \circ \text{can} = \text{can}'$. Also one may easily check that Ξ is bijective, with inverse given by

$$\Xi^{-1}(a \otimes h^*) = \sum_{i=1}^n e_i \cdot a \otimes (f^{(-1)1} \rightharpoonup h^* S^{-1} \leftarrow V_L^1)(f^{(-1)2} \rightharpoonup e^i \leftarrow V_L^2)$$

where f is the gauge element from (2.3) and V_L is the element introduced by (2.22).

Remark 3.2.3 We proved in **Proposition 2.2.3** that the functor $(-)^H : A \# H \mathcal{M} \longrightarrow_B \mathcal{M}$ has a left adjoint, namely $A \otimes_B (-)$. The counit of this adjunction is $\varepsilon_M : A \otimes_B M^H \longrightarrow M$, $\varepsilon_M(a \otimes_B m) = am$ (for M a left $A \# H$ -module); it is left $A \# H$ -linear. If ε_M is an isomorphism for all modules M , we call this the **Weak Structure Theorem**.

Remark 3.2.4 Let $\theta_t : H^* \longrightarrow H$ be the isomorphism of the **Corollary 2.1.5**, $\theta_t(h^*) = h^*(q_L^1 t_1 p_L^1) q_L^2 t_2 p_L^2$. As in the classical Hopf case, we get the relation $(I_A \otimes \theta_t) \circ \text{can} = [-, -]$ which indicates that the Morita map $[-, -]$ and the Galois map can will be simultaneously bijective:

$$\begin{aligned} (I_A \otimes \theta_t) \circ \text{can}(a \otimes_B b) &= (I_A \otimes \theta_t) \left(\sum_{i=1}^n (p_R^1 \cdot a)(p_R^2 e_i \cdot b) \otimes e^i \right) \\ (2.1.5) &= \sum_{i=1}^n (p_R^1 \cdot a)(p_R^2 e_i \cdot b) \otimes e^i (q_L^1 t_1 p_L^1) q_L^2 t_2 p_L^2 \\ &= (p_R^1 \cdot a)(p_R^2 q_L^1 t_1 p_L^1 \cdot b) \otimes q_L^2 t_2 p_L^2 \\ (2.8) &= (x^1 \cdot a)(x^2 \beta S(x^3) q_L^1 t_1 p_L^1 \cdot b) \otimes q_L^2 t_2 p_L^2 \\ (2.31) &= (x^1 \cdot a)(x^2 \beta q_L^1 t_1 p_L^1 \cdot b) \otimes x^3 q_L^2 t_2 p_L^2 \\ (2.33) &= (x^1 \cdot a)(x^2 t_1 p_L^1 \cdot b) \otimes x^3 t_2 p_L^2 \\ (2.36) &= (a \# t)(p_L^1 \cdot b \# p_L^2) \\ &= [a, b] \end{aligned}$$

Now we have all the ingredients to prove the analogue of **Theorem 3.1** of [2] in case of a finite dimensional quasi-Hopf algebra and the proof follows closely the one in [2].

Theorem 3.2.5 Let H be a finite dimensional quasi-Hopf algebra, A a left H -module algebra and $B = A^H$ the subalgebra of H -invariants. Then the following statements are equivalent:

1. The extension A/B is Galois;
2. The map $\text{can} : A \otimes_B A \longrightarrow A \otimes H^*$ is surjective;
3. The Morita map $[-, -]$ is bijective;
4. The Morita map $[-, -]$ is surjective;
5. The Weak Structure Theorem holds for ${}_A(H\mathcal{M})$;
6. The counit of the adjunction $\varepsilon_M : A \otimes_B M^H \longrightarrow M$ is surjective for all left $A\#H$ -modules M ;
7. A is a generator for the category ${}_A(H\mathcal{M}) \simeq {}_{A\#H}\mathcal{M}$.

Proof. 1. \implies 2., 3. \implies 4., 5. \implies 6. Obviously.

1. \iff 3., 2. \iff 4. Come from the previous remark.

4. \implies 3. It results from the classical Morita theory for rings.

4. \implies 5. The injectivity of the counit of the adjunction: let $\sum_i a_i \otimes_B m_i \in A \otimes_B M^H$ such that $\sum_i a_i m_i = 0$. By the surjectivity of $[-, -]$ we may find $\sum_k c_k \otimes_B d_k \in A \otimes_B A$ such that $\sum_k [c_k, d_k] = 1_A \# 1_H$ (which means $\sum_k (x^1 \cdot c_k)(x^2 t_1 p_L^1 \cdot d_k) \# (x^3 t_2 p_L^2) = 1_A \# 1_H$). We get then

$$\begin{aligned}
\sum_i a_i \otimes_B m_i &= \sum_i (1 \# 1) a_i \otimes_B m_i \\
&= \sum_{i,k} [c_k, d_k] a_i \otimes_B m_i \\
&= \sum_{i,k} c_k (d_k, a_i) \otimes_B m_i \\
&= \sum_{i,k} c_k \otimes_B (d_k, a_i) m_i \\
(3.3) \quad &= \sum_{i,k} c_k \otimes_B \{ t [(p_L^1 \cdot d_k)(p_L^2 \cdot a_i)] \} m_i \\
&= \sum_{i,k} c_k \otimes_B t \{ [(p_L^1 \cdot d_k)(p_L^2 \cdot a_i)] m_i \} \\
&= \sum_{i,k} c_k \otimes_B t \{ (X^1 p_L^1 \cdot d_k) [(X^2 p_L^2 \cdot a_i)(X^3 m_i)] \} \\
&= \sum_{i,k} c_k \otimes_B t \{ (p_L^1 \cdot d_k) [(p_L^2 \cdot a_i) m_i] \} \\
&= \sum_{i,k} c_k \otimes_B t \{ (p_L^1 \cdot d_k) [p_L^2 \cdot (a_i m_i)] \} \\
&= 0
\end{aligned}$$

where in the last four lines we used that $m_i \in M^H$. For the surjectivity, let $m \in M$. Then we compute that

$$\begin{aligned}
\varepsilon_M(\sum_k c_k \otimes_B t [(p_L^1 \cdot d_k)(p_L^2 m)]) &= \sum_k c_k \{ t [(p_L^1 \cdot d_k)(p_L^2 m)] \} \\
&= \sum_k \underbrace{[(x^1 \cdot c_k)(x^2 t_1 p_L^1 \cdot d_k)]}_{1_A \# 1_H} (x^3 t_2 p_L^2 m) \\
&= m
\end{aligned}$$

proving that ε_M is indeed surjective.

5. \implies 3., 6. \implies 4. $A\#H$ is an $A\#H$ -module with the action given by the algebra multiplication, so by hypothesis $\varepsilon_{A\#H}$ is bijective (respectively surjective). We get the following sequence of bijections (respectively surjections)

$$A \otimes_B A \simeq A \otimes_B \left(\int_H^l \otimes A \right) = A \otimes_B (H \otimes A)^H \simeq A \otimes_B (A\#H)^H \xrightarrow{\varepsilon_{A\#H}} A\#H$$

Explicitly, this means

$$\begin{aligned} a \otimes_B b &\longrightarrow a \otimes_B (t \otimes b) \longrightarrow a \otimes_B (t_1 p_L^1 \cdot b \# t_2 p_L^2) \\ &\longrightarrow a \cdot (t_1 p_L^1 \cdot b \# t_2 p_L^2) = (a \# 1)(t_1 p_L^1 \cdot b \# t_2 p_L^2) \\ &= (x^1 \cdot a)(x^2 t_1 p_L^1 \cdot b) \# x^3 t_2 p_L^2 \\ &= [a, b] \end{aligned}$$

hence the Morita map $[-, -]$ is bijective (respectively surjective).

5. \implies 7, 7. \implies 6. The proofs are identical to those in the Hopf case, so we omit them. \blacksquare

We give now two examples of Galois extensions.

First, let $F \in H \otimes H$ be a gauge transformation. If we denote by H_F the quasi-Hopf algebra obtained by twisting the comultiplication of H via F , then in [8] it is proven that there is a new multiplication on A , namely $a \circ b = (F^{(-1)1} \cdot a)(F^{(-1)2} \cdot b)$, for $a, b \in A$ such that A , with this new multiplication (denoted $A_{F^{-1}}$), becomes a left H_F -module algebra. In this case the categories ${}_A(H\mathcal{M})$ and ${}_{A_{F^{-1}}}(H_F\mathcal{M})$ are isomorphic and there is an algebra isomorphism between the smash products $A\#H$ and $A_{F^{-1}}\#H_F$, which sends $a\#h \longrightarrow F^1 \cdot a\#F^2 h$, for all $a \in A$, $h \in H$.

Remark also that B , the space of H -invariants, remains the same, as the action of H is not modified. Moreover, it is an associative algebra with the multiplication induced by the new multiplication on $A_{F^{-1}}$, as H acts trivially on B and F is a gauge transformation. As the categories ${}_A(H\mathcal{M})$ and ${}_{A_{F^{-1}}}(H_F\mathcal{M})$ are isomorphic and the following diagram of functors is commutative

$$\begin{array}{ccc} {}_A(H\mathcal{M}) & \simeq & {}_{A_{F^{-1}}}(H_F\mathcal{M}) \\ A \otimes_B (-) \swarrow \searrow (-)^H & & (-)^{H_F} \nearrow \nwarrow A_{F^{-1}} \otimes_B - \\ & B\mathcal{M} & \end{array}$$

as it can be easily checked, the counits of these two adjunctions will be simultaneously bijective. Hence, by **Theorem 3.2.5**, $B \subseteq A$ is Galois $\Leftrightarrow B \subseteq A_{F^{-1}}$ is Galois.

For the second example, let \mathcal{A} be a right H -comodule algebra, as it was defined in [15]. That is, \mathcal{A} is an associative algebra endowed with an algebra morphism $\rho : \mathcal{A} \longrightarrow \mathcal{A} \otimes H$ and an invertible element $\phi_\rho \in \mathcal{A} \otimes H \otimes H$, such that

$$\phi_\rho(\rho \otimes I)\rho(a)\phi_\rho^{-1} = (I \otimes \Delta)\rho(a) \quad (3.5a)$$

$$(I \otimes \varepsilon)\rho(a) = a \quad (3.5b)$$

$$(I \otimes I \otimes \Delta)(\phi_\rho)(\rho \otimes I \otimes I)(\phi_\rho) = (1_{\mathcal{A}} \otimes \phi)(I \otimes \Delta \otimes I)(\phi_\rho)(\phi_\rho \otimes 1) \quad (3.5c)$$

$$(I \otimes \varepsilon \otimes I)(\phi_\rho) = (I \otimes I \otimes \varepsilon)(\phi_\rho) = 1 \otimes 1 \quad (3.5d)$$

hold for all $a \in \mathcal{A}$. We shall denote $\phi_\rho = X_\rho^1 \otimes X_\rho^2 \otimes X_\rho^3$ and $\phi_\rho^{-1} = x_\rho^1 \otimes x_\rho^2 \otimes x_\rho^3$. Following [6], we may define the quasi-smash product $\mathcal{A}\overline{\#}H^*$. As vector space, this is $\mathcal{A} \otimes H^*$ endowed with a multiplication given by

$$(a\overline{\#}h^*)(b\overline{\#}g^*) = ab_0 x_\rho^1 \overline{\#}(h^* \leftarrow b_1 x_\rho^2)(g^* \leftarrow x_\rho^3) \quad (3.6)$$

for any $a, b \in \mathcal{A}$, $h^*, g^* \in H^*$. Using the left H -action given by

$$g(a\#h^*) = a\#g \rightharpoonup h^*$$

for any $g \in H$, $a \in \mathcal{A}$, $h^* \in H^*$, the quasi-smash product $\mathcal{A}\#H^*$ becomes a left H -module algebra with invariants $(\mathcal{A}\#H^*)^H = \mathcal{A}\#k\varepsilon \simeq \mathcal{A}$. For this module algebra, the Galois map can is

$$\begin{aligned} can & : (\mathcal{A}\#H^*) \otimes_{\mathcal{A}} (\mathcal{A}\#H^*) \longrightarrow (\mathcal{A}\#H^*) \otimes H^* \\ (a\#h^*) \otimes_{\mathcal{A}} (b\#g^*) & \longrightarrow \sum_{i=1}^n (a\#p_R^1 \rightharpoonup h^*)(b\#p_R^2 e_i \rightharpoonup g^*) \otimes e^i \end{aligned}$$

where $a, b \in \mathcal{A}$, $h^*, g^* \in H^*$. But $\mathcal{A}\#H^* = (\mathcal{A}\#\varepsilon)(1_{\mathcal{A}}\#H^*)$, meaning that it's enough to consider elements of the type $(a\#h^*) \otimes_{\mathcal{A}} (1_{\mathcal{A}}\#g^*)$, as in the Hopf case. Now the formula for the Galois map becomes

$$can((a\#h^*) \otimes_{\mathcal{A}} (1_{\mathcal{A}}\#g^*)) = \sum_{i=1}^n (a\#p_R^1 \rightharpoonup h^*)(1_{\mathcal{A}}\#p_R^2 e_i \rightharpoonup g^*) \otimes e^i$$

We need the following element introduced in [15]: $q_\rho = q_\rho^1 \otimes q_\rho^2 = X_\rho^1 \otimes S^{-1}(\alpha X_\rho^3) X_\rho^2$. This element has similar properties with q_R :

$$(1_{\mathcal{A}} \otimes S^{-1}(a_1))q_\rho \rho(a_0) = (a \otimes 1)q_\rho \quad (3.7)$$

$$(\rho \otimes I)(q_R)\phi_\rho^{-1} = (1_{\mathcal{A}} \otimes 1 \otimes S^{-1}(X_\rho^3))(X_\rho^1 \otimes q_\rho \Delta(X_\rho^2)) \quad (3.8)$$

for all $a \in \mathcal{A}$.

Proposition 3.2.6 *The quasi-smash product $\mathcal{A}\#H^*$ is a Galois extension of \mathcal{A} , with inverse of the Galois map given by*

$$can^{-1}((a\#h^*) \otimes g^*) = \sum_{i=1}^n (a\#h^*)(q_\rho^1 \# e^i S \leftarrow q_\rho^2) \otimes_{\mathcal{A}} (1_{\mathcal{A}}\#g^* \leftarrow e_i)$$

for all $a \in \mathcal{A}$, $h^*, g^* \in H^*$.

Proof. For all $a \in \mathcal{A}$ and $h^*, g^* \in H^*$, we compute that

$$\begin{aligned} can^{-1} \circ can((a\#h^*) \otimes_{\mathcal{A}} (1_{\mathcal{A}}\#g^*)) & = can^{-1}(\sum_{i=1}^n (a\#p_R^1 \rightharpoonup h^*)(1_{\mathcal{A}}\#p_R^2 e_i \rightharpoonup g^*) \otimes e^i) \\ & = \sum_{i,j=1}^n [(a\#p_R^1 \rightharpoonup h^*)(1_{\mathcal{A}}\#p_R^2 e_i \rightharpoonup g^*)](q_\rho^1 \# e^j S \leftarrow q_\rho^2) \otimes_{\mathcal{A}} (1_{\mathcal{A}}\#e^i \leftarrow e_j) \\ & = \sum_{i,j=1}^n (a\#X^1 p_R^1 \rightharpoonup h^*)[(1_{\mathcal{A}}\#X^2 p_R^2 e_j e_i \rightharpoonup g^*)(q_\rho^1 \# X^3 \rightharpoonup e^j S \leftarrow q_\rho^2)] \\ & \quad \otimes_{\mathcal{A}} (1_{\mathcal{A}}\#e^i) \\ & = \sum_{i,j=1}^n (a\#X^1 p_R^1 \rightharpoonup h^*)[(1_{\mathcal{A}}\#X^2 p_R^2 S(X^3) e_j S(q_\rho^2) e_i \rightharpoonup g^*)(q_\rho^1 \# e^j S)] \\ & \quad \otimes_{\mathcal{A}} (1_{\mathcal{A}}\#e^i) \end{aligned}$$

$$\begin{aligned}
(2.8) &= \sum_{i,j=1}^n (a\overline{\#}h^*)[(1_{\mathcal{A}}\overline{\#}\beta e_j S(q_\rho^2)e_i \rightharpoonup g^*)(q_\rho^1\overline{\#}e^j S)] \otimes_{\mathcal{A}} (1_{\mathcal{A}}\overline{\#}e^i) \\
(3.6) &= \sum_{i,j=1}^n (a\overline{\#}h^*)(q_{\rho_0}^1 x_\rho^1 \overline{\#}(\beta e_j S(q_\rho^2)e_i \rightharpoonup g^* \leftarrow q_{\rho_1}^1 x_\rho^2)(e^j S \leftarrow x_\rho^3)) \\
&\quad \otimes_{\mathcal{A}} (1_{\mathcal{A}}\overline{\#}e^i) \\
&= \sum_{i,j=1}^n (a\overline{\#}h^*)(q_{\rho_0}^1 x_\rho^1 \overline{\#}(\beta e_j S(x_\rho^3)S(q_\rho^2)e_i \rightharpoonup g^* \leftarrow q_{\rho_1}^1 x_\rho^2)(e^j S)) \\
&\quad \otimes_{\mathcal{A}} (1_{\mathcal{A}}\overline{\#}e^i) \\
(3.5c) &= \sum_{i,j=1}^n (a\overline{\#}h^*)(x_\rho^1 X_\rho^1 \overline{\#}(\beta e_j S(x_{\rho_1}^3 X^2 X_{\rho_2}^2)\alpha x_{\rho_2}^3 X^3 X_\rho^3 e_i \rightharpoonup g^* \\
&\quad \leftarrow x_\rho^2 X^1 X_{\rho_1}^2)(e^j S)) \otimes_{\mathcal{A}} (1_{\mathcal{A}}\overline{\#}e^i) \\
(2.2a), (3.5d) &= \sum_{i,j=1}^n (a\overline{\#}h^*)(X_\rho^1 \overline{\#}(\beta e_j S(X^2 X_{\rho_2}^2)\alpha X^3 X_\rho^3 e_i \rightharpoonup g^* \\
&\quad \leftarrow X^1 X_{\rho_1}^2)(e^j S)) \otimes_{\mathcal{A}} (1_{\mathcal{A}}\overline{\#}e^i) \\
(2.2b) &= \sum_{i=1}^n (a\overline{\#}h^*)(X_\rho^1 \overline{\#}g^*(X^1 X_{\rho_1}^2 \beta S(X^2 X_{\rho_2}^2)\alpha X^3 X_\rho^3 e_i)\varepsilon) \\
&\quad \otimes_{\mathcal{A}} (1_{\mathcal{A}}\overline{\#}e^i) \\
(2.2b) &= \sum_{i=1}^n (a\overline{\#}h^*)(1_{\mathcal{A}}\overline{\#}g^*(X^1 \beta S(X^2)\alpha X^3 e_i)\varepsilon) \otimes_{\mathcal{A}} (1_{\mathcal{A}}\overline{\#}e^i) \\
(2.2c) &= \sum_{i=1}^n (a\overline{\#}h^*)(1_{\mathcal{A}}\overline{\#}g^*(e_i)\varepsilon) \otimes_{\mathcal{A}} (1_{\mathcal{A}}\overline{\#}e^i) \\
&= \sum_{i=1}^n (a\overline{\#}h^*) \otimes_{\mathcal{A}} (1_{\mathcal{A}}\overline{\#}g^*)
\end{aligned}$$

Next, we have that

$$\begin{aligned}
can \circ can^{-1}((a\overline{\#}h^*) \otimes g^*) &= can(\sum_{j=1}^n (a\overline{\#}h^*)(q_\rho^1 \overline{\#} e^j S \rightarrow q_\rho^2) \otimes_A (1_A \overline{\#} g^* \leftarrow e_j)) \\
(3.6) &= can(\sum_{j=1}^n (aq_{\rho_0}^1 x_\rho^1 \overline{\#} (h^* \leftarrow q_{\rho_1}^1 x_\rho^2)(e^j S \rightarrow q_\rho^2 x_\rho^3)) \otimes_A (1_A \overline{\#} g^* \leftarrow e_j)) \\
&= \sum_{i,j=1}^n [aq_{\rho_0}^1 x_\rho^1 \overline{\#} (p_{R_1}^1 \rightarrow h^* \leftarrow q_{\rho_1}^1 x_\rho^2)(p_{R_2}^1 \rightarrow e^j S \rightarrow q_\rho^2 x_\rho^3)](1_A \overline{\#} p_R^2 e_i \rightarrow g^* \\
&\leftarrow e_j) \otimes e^i \\
(3.6) &= \sum_{i,j=1}^n (aq_{\rho_0}^1 x_\rho^1 y_\rho^1 \overline{\#} [(p_{R_1}^1 \rightarrow h^* \leftarrow q_{\rho_1}^1 x_\rho^2 y_{\rho_1}^2)(p_{R_2}^1 \rightarrow e^j S \rightarrow q_\rho^2 x_\rho^3 y_{\rho_2}^2)](p_R^2 e_i \rightarrow g^* \\
&\leftarrow e_j y_\rho^3)) \otimes e^i \\
(3.5c) &= \sum_{i,j=1}^n (ax_\rho^1 \overline{\#} (X^1 p_{R_1}^1 \rightarrow h^* \leftarrow x_\rho^2)[(X^2 p_{R_2}^1 \rightarrow e^j S \rightarrow S^{-1}(\alpha)x_{\rho_1}^3)(X^3 p_R^2 e_i \rightarrow g^* \\
&\leftarrow e_j x_{\rho_2}^3)]) \otimes e^i \\
(2.2a) &= \sum_{i,j=1}^n (a\overline{\#} (X^1 p_{R_1}^1 \rightarrow h^*)[(X^2 p_{R_2}^1 \rightarrow e^j S)(X^3 p_R^2 e_i \rightarrow g^* \leftarrow e_j \alpha)]) \otimes e^i \\
&= \sum_{i=1}^n a\overline{\#} (X^1 p_{R_1}^1 \rightarrow h^*) \otimes g^*(S(X^2 p_{R_2}^1) \alpha X^3 p_R^2 e_i) e^i \\
(2.17) &= a\overline{\#} h^* \otimes g^*
\end{aligned}$$

■

3.2.2 Total integrals

Definition 3.2.7 Let H be a quasi-Hopf algebra and A a left H -module algebra. A **total integral** for A is a left H -morphism $\Phi : H^* \longrightarrow A$ such that $\Phi(\varepsilon) = 1_A$ (on H^* we take the structure of left H -module given by translation: $(h \rightharpoonup h^*)(g) = h^*(gh)$ for all $h^* \in H^*$, $g, h \in H$).

Proposition 3.2.8 With notations as above, the following statements are equivalent:

1. The Morita map $(-, -)$ is surjective;
2. There is a total integral for A ;
3. A has an element of trace one (i.e. $a \in A$ such that $t \cdot a = 1_A$).

Proof. $1. \implies 2.$ Let $\sum_i a_i \otimes_B b_i \in A \otimes_B A$ such that $\sum_i (a_i, b_i) = 1_A$. We define $\Phi : H^* \longrightarrow A$,

$$\Phi(h^*) = \sum_i q_L^1 t_1 \cdot [(p_L^1 \cdot a_i)(p_L^2 \cdot b_i)] h^* S(q_L^2 t_2)$$

Then

$$\begin{aligned}
\Phi(\varepsilon) &= \sum_i q_L^1 t_1 \cdot [(p_L^1 \cdot a_i)(p_L^2 \cdot b_i)] \varepsilon(q_L^2 t_2) \\
(2.7) &= \sum_i \alpha t \cdot [(p_L^1 \cdot a_i)(p_L^2 \cdot b_i)] \\
&= t \cdot [(p_L^1 \cdot a_i)(p_L^2 \cdot b_i)] \\
&= \sum_i (a_i, b_i) = 1_A
\end{aligned}$$

and

$$\begin{aligned}
\Phi(h \rightharpoonup h^*) &= \sum_i q_L^1 t_1 \cdot [(p_L^1 \cdot a_i)(p_L^2 \cdot b_i)] h^* S(S^{-1}(h) q_L^2 t_2) \\
(2.31) &= \sum_i h q_L^1 t_1 \cdot [(p_L^1 \cdot a_i)(p_L^2 \cdot b_i)] h^* S(q_L^2 t_2) \\
&= h \cdot \Phi(h^*)
\end{aligned}$$

2. \implies 1. Let Φ be a total integral. As $t \neq 0$ and H is finite dimensional, we may find $h^* \in H^*$ such that $h^*(t) = 1$. Then $(t \rightharpoonup h^*)(h) = h^*(ht) = \varepsilon(h)h^*(t) = \varepsilon(h)$, $\forall h \in H$, meaning that $t \rightharpoonup h^* = \varepsilon$. Hence

$$\begin{aligned}
(1_A, \Phi(h^*)) &= t \cdot [(p_L^1 \cdot 1_A)(p_L^2 \cdot \Phi(h^*))] \\
&= t \cdot \Phi(h^*) \\
&= \Phi(t \rightharpoonup h^*) \\
&= \Phi(\varepsilon) = 1_A
\end{aligned}$$

and using the B -bilinearity of $(-, -)$ we get the surjectivity.

2. \implies 3. Let Φ be a total integral. As above, consider $h^* \in H^*$ such that $t \rightharpoonup h^* = \varepsilon$. Then $a = t \cdot \Phi(h^*)$ is a trace one element.

3. \implies 2. Let $a \in A$ an element of trace one and $h^* \in H^*$, $t \rightharpoonup h^* = \varepsilon$ as above. Define $\Phi : H^* \longrightarrow A$, $\Phi(g^*) = q_L^1 t_1 \cdot a g^* S(q_L^2 t_2)$. Then $\Phi(\varepsilon) = q_L^1 t_1 \cdot a \varepsilon S(q_L^2 t_2) = \alpha t \cdot a = t \cdot a = 1_A$ and $\Phi(h \rightharpoonup g^*) = q_L^1 t_1 \cdot a g^* S(S^{-1}(h) q_L^2 t_2) = h q_L^1 t_1 \cdot a g^* S(q_L^2 t_2) = h \Phi(g^*)$, which means that Φ is a total integral. ■

Example 3.2.9 1) Let $F \in H \otimes H$ be a gauge transformation, as in the example of the previous section. Then a total integral for A remains a total integral for $A_{F^{-1}}$, as the action of H is not modified. So the Morita maps $(-, -)$ will be simultaneously bijective.

2) Let \mathcal{A} be a right H -comodule algebra. Then it is easy to see that the map $\Phi : H^* \longrightarrow \mathcal{A} \# H^*$, given by $\Phi(h^*) = 1 \# h^*$, is a total integral. Hence in this example we get the equivalence of categories $\mathcal{A} \mathcal{M} \simeq_{(\mathcal{A} \# H^*) \# H} \mathcal{M}$

In connection with the notion of total integral, Bulacu and Nauwelaerts proved in [7] the following three statements, for a dual quasi-Hopf algebra and a comodule algebra. But in the finite dimensional case this is the same as working with the quasi-Hopf algebra and a module algebra, so we state them for completeness:

Proposition 3.2.10 (Proposition 2.9, [7]) *If A is a left H -module algebra and there is a total integral $\Phi : H^* \longrightarrow A$, then each relative module $M \in ({}_H \mathcal{M})_A$ is injective as an H -module (where $({}_H \mathcal{M})_A$ is the category of right A -modules in the monoidal category ${}_H \mathcal{M}$).*

Corollary 3.2.11 (Corollary 2.10, [7]) *Under the previous hypotheses, the following statements are equivalent:*

1. A is an injective left H -module;

2. There is a total integral on A ;
3. Each object in $({}_H\mathcal{M})_A$ is injective as an H -module.

Theorem 3.2.12 (Theorem 2.11, [7]) *If A is a left H -module algebra and there is a total integral $\Phi : H^* \longrightarrow A$ which is multiplicative, then for every $M \in ({}_H\mathcal{M})_A$ the counit $\varepsilon_M : A \otimes_B M^H \longrightarrow M$, $\varepsilon_M(a \otimes_B m) = am$, of the adjunction $\mathcal{M}_B \xrightleftharpoons[(-)^H]{- \otimes_B A} ({}_H\mathcal{M})_A$ is an isomorphism.*

As remarked in the quoted paper, working with left or right A -modules is essentially the same, just by passing to the opposite algebra A^{op} (which is a module algebra over $H^{op, cop}$), so we can rephrase these results in our context and obtain the following theorem:

Theorem 3.2.13 *Let H be a finite dimensional quasi-Hopf algebra, A a left H -module algebra and $B = A^H$ the subalgebra of H -invariants. Then the following statements are equivalent:*

1. The Morita map $(-, -)$ is surjective;
2. There is a total integral for A ;
3. A has an element of trace one (i.e. $a \in A$ such that $t \cdot a = 1_A$);
4. A is an injective left H -module;
5. Each object in ${}_A({}_H\mathcal{M})$ is injective as an H -module.

Combining the results of the previous two sections, we may state now the following:

Theorem 3.2.14 *Let H be a finite dimensional quasi-Hopf algebra, A a left H -module algebra and $B = A^H$ the subalgebra of H -invariants. Then the following statements are equivalent:*

1. The functors ${}_B\mathcal{M} \xrightleftharpoons[(-)^H]{A \otimes_B (-)} {}_A({}_H\mathcal{M})$ are a pair of inverse equivalences (**Strong Structure Theorem**);
2. The Morita maps $[,]$ and $(,)$ are surjective;
3. The extension $B \subseteq A$ is Galois and there is a total integral on A .

Conclusion 3.2.15 *As noticed in the introduction, if we want to start with a right H -comodule algebra A and take the usual definition for coinvariants \mathcal{A}^{coH} , this does not work any more in the quasi-Hopf setting. For example, if we take a left H -module algebra A and form the smash product $A \# H$, then this is a right comodule algebra ([8]), but we cannot recover A from the coinvariants as in the Hopf case (we get something bigger). There are two possible ways to overcome this problem: either to find an adequate definition for the coinvariants, as it was done in [23], or to pass to bicategories. Anyway, if we want the previous example to fit, we need some coinvariants which are associative in the category of left H -modules, so we can only get a Morita context in this monoidal category. It would be interesting to see which are the connections between these two types of Morita contexts, knowing that in the Hopf case these two contexts are the same.*

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